Monodromy of the hypergeometric differential equation of type (k, n)

Sasaki, Takeshi; Takayama, Nobuki; Yoshida, Masaaki; Matsumoto, Keiji

数理解析研究所講究録
第 816 巻 1992 年 121-140

Monodromy of the hypergeometric differential equation of type (k,n)

神戸大 理　佐々木 武 (Takeshi Sasaki), 高山 信毅 (Nobuki Takayama)
九大 理　吉田 正章 (Masaaki Yoshida), 松本 圭司 (Keiji Matsumoto)

Abstract

In this paper we present a set of generators of the monodromy group of the hypergeometric differential equation of type (k,n). Since fundamental solutions can be expressed by integrals of products of complex powers of linear forms, it might not be impossible to find the monodromy representation of the system by tracing the change of cycles of integration ([Aom]). But, if one wants to study properties of the monodromy group, it is essential to know nice generators explicitly; this is the very thing we do in this paper.

Contents

0. Introduction
1. The configuration space X, the submanifold Q and a base arrangement
2. Twisted cycles and a basis of solutions
3. Circuit matrix $M(1,\dots,r+1;\alpha)$
4. Relation between $E(r+1,n+1;\alpha)$ and $E(2,n+1;\alpha')$
5. Action of the braid group B_{n+1} on a collection of solutions of $E(2,n+1;\alpha)$
6. Generators
References

Introduction

Fix positive integers r and $n\,(\geq r+1)$, and complex numbers $\alpha_1,\dots,\alpha_n$ such that

$$\alpha_1,\dots,\alpha_n,\ \alpha_1+\cdots+\alpha_n\notin\mathbb{Z}.$$

Let $L_j\,(1\leq j\leq n)$ be linear forms in $t=(t_0=1,t_1,\dots,t_r)\in\mathbb{C}^r$:

$$L_j=\sum_{i=0}^{r}x_{ij}t_i,$$

where $x=(x_{ij})$ are complex variables such that any $(r+1)\times(r+1)$-minor of the matrix

$$\begin{pmatrix}1 & x_{01} & \cdots & x_{0n}\\ 0 & x_{11} & \cdots & x_{1n}\\ \vdots & \vdots & \cdots & \vdots\\ 0 & x_{r1} & \cdots & x_{rn}\end{pmatrix}$$

does not vanish. The set of integrals

$$\int \prod_j L_j^{\alpha_j - 1} dt_1 \wedge \cdots \wedge dt_r$$

over various cycles, as a set of functions of x, spans an

$$\binom{n-1}{r}$$

dimensional linear space, which is known to be the solution space of the hypergeometric differential equation

$$E(r+1, n+1; \alpha) = E(r+1, n+1; \alpha_0, \alpha_1, \ldots, \alpha_n)$$

of type $(r+1, n+1)$, where α_0 is determined by

$$\alpha_0 + \cdots + \alpha_n = n - r.$$

We regard x as a variable representing points of the configuration space $X = X(r+1, n+1)$ of $n+1$ hyperplanes in the r-dimensional projective space:

$$X(r+1, n+1) = GL(r+1, \mathbb{C}) \backslash M^*(r+1, n+1) / H(n+1),$$

where the symbols used are defined in Section 1. We fix a base point $\dot{x} \in X$ and a basis of solutions at the point; we continue analytically the solutions along possible paths in X, which causes a linear change of the basis; the totality of such linear transformations forms a group called the monodromy group with respect to the basis. Our goal is to find explicit matrices corresponding to a set of generators of the fundamental group of X.

By the way, the hypergeometric differential equation of type $(2, n+1)$ is known by the name of Appell-Lauricella's hypergeometric equation in $n-2$ variables; it is especially simple since the integral representation above is of 1-dimensional. The monodromy of this system is a representation of the colored braid group, which is well studied, while we shall use for our purpose the 1-cocycle representation of the braid group associated to the system. The key to relate this system to our system $E(r+1, n+1; \alpha)$ is the following fact due to [Ter]: when $x \in X$ defines $n+1$ hyperplanes in the r-dimensional projective space such that the $n+1$ points dual to the hyperplanes are on a nonsingular curve of degree r in the dual projective space, then the system $E(r+1, n+1; \alpha)$ boils down to the r-wedge product of the system $E(2, n+1; \alpha') = E(2, n+1; \alpha'_0, \ldots, \alpha'_n)$ where $\alpha_j - \alpha'_j \in \mathbb{Z}$ and

$$\alpha'_0 + \cdots + \alpha'_n = n - 1.$$

Let Q be the submanifold of X consisting of such x then the above fact can be symbolized as follows:

$$E(r+1, n+1; \alpha) \mid_Q = \wedge^r E(2, n+1; \alpha'), \quad \alpha - \alpha' \in \mathbb{Z}^n.$$

Let $\rho(j_1,\ldots,j_{r+1})$ be a loop in X with base point $\overset{\bullet}{x}$ which is described by the following move of hyperplanes H_j $(0\leq j\leq n)$: let us choose one index among $j_1,\ldots,j_{r+1}$ and call it j'; all the hyperplanes but $H_{j'}$ do not move, the hyperplane approaches sufficiently near to the intersection point of the r remaining hyperplanes among $r+1$ hyperplanes $H_{j_1},\ldots,H_{j_{r+1}}$, goes once around the point in the positive sense, and travels back to the original position tracing back the previous route. The choice of the loop $\rho(j_1,\ldots,j_{r+1})$ is by no means unique but anyhow these generate the fundamental group of X.

We choose the base point $\overset{\bullet}{x}\in X$ so that the corresponding n hyperplanes are defined over reals; the complement of the n hyperplanes has $\binom{n-1}{r}$ compact chambers; the integrals of the r-form above on these chambers give a set of independent solutions, which we take as a basis. We choose the base point carefully (Section 1) in $Q\subset X$ so that the hyperplanes $H_1,\ldots,H_{r+1}$ bound a simplex in real affine t-space, and choose the loop $\rho(1,\ldots,r+1)$ so that during the journey of a hyperplane $H_{j'}$ $(j'=1,\ldots,r$ or $r+1)$, the simplex remains to be a small simplex. It is to be noted that the homotopy class of the loop does not depend on the choice of the index j', and that this move of the point can not be done inside Q, it must travel beyond Q in X. The linear change $M(1,\ldots,r+1;\alpha)$ of the basis caused by the loop $\rho(1,\ldots,r+1)$, which will be called the circuit matrix, is obtained in Section 3. In order to describe the change of the basis caused by another loop $\rho(1,\ldots,r,r+2)$, we first exchange the two hyperplanes H_{r+1} and H_{r+2}, which can be done by a move of x in Q, next apply $M(1,\ldots,r+1;\alpha)$, of course α_{r+1} must be read α_{r+2}, and then exchange again the two hyperplanes H_{r+1} and H_{r+2}. Since the process of exchange can be done inside Q, by virtue of the relation between $E(r+1,n+1)$ and $E(2,n+1)$, we can describe it in terms of the change of 1-dimensional cycles. In the same way, by applying successively the process to the standard generators of the braid group B_{n+1}, we can find circuit matrices $M(j_1,\ldots,j_{r+1};\alpha)$ corresponding to the other loops $\rho(j_1,\ldots,j_{r+1})$.

In [MSY], a set of generators of $E(3,6;1/2,\ldots,1/2)$ is obtained by the use of periods of a family of K3 surfaces; in this paper, we treat general k, general n and general α. Our generators obtained in this paper have good properties (e.g. all but one eigenvalues are 1) which make closer studies of the monodromy group possible; see our forthcoming paper [MSTY].

Acknowledgement. The authors are grateful to Professor Kita, who kindly imformed them the result of [Kit], which guarantees the validity of the main theorem (Section 6) under the weakest possible condition: $\alpha_0,\ldots,\alpha_n\notin\mathbb{Z}$.

1. The configuration space X, the submanifold Q and a base arrangement

In this section we introduce the configuration space $X=X(r+1,n+1)$ of the $n+1$ hyperplanes in general position in the r-dimensional complex projective space P^r, define a submanifold Q, and choose a point $\overset{\bullet}{x}$ in Q which shall be used as a base point.

Let $t_0:\cdots:t_r$ be a system of homogeneous coordinates of the projective space, and consider $n+1$ hyperplanes $H_j(x)$, called an arrangement, defined by linear equations

$$L_j(x):=\sum_{i=0}^{r}x_{ij}t_i=0,\quad 0\leq j\leq n.$$

These hyperplanes are said to be of general position if no $r+1$ planes meet at a point, or equivalently, if any $(r+1)\times(r+1)$-minor of the matrix $x=(x_{ij})$ does not vanish. Two such arrangements are considered to be equivalent if one is sent to the other by a projective transformation of P^r. Thus the space of (equivalence classes of) such arrangements are given by the double quotient space

$$X = X(r+1,n+1) = GL(r+1,\mathbf{C})\backslash M^*(r+1,n+1)/H(n+1),$$

where $M^*(r+1,n+1)$ is the space of $(r+1)\times(n+1)$-matrices $x=(x_{ij})$ that any $(r+1)\times(r+1)$-minor does not vanish, and $H(n+1)$ is the subgroup consisting of diagonal matrices in $GL(n+1,\mathbf{C})$. This space X, which has the natural structure of an $r(n-r-1)$-dimensional affine manifold, is called the configuration space of $n+1$ hyperplanes (in general position) in P^r.

Let Q be the $(n-2)$-dimensional submanifold of X consisting of the arrangements such that there is a nonsingular curve of degree r along which the $n+1$ hyperplanes osculate, or equivalently, that there is a nonsingular curve of degree r in the dual projective space on which the $n+1$ points dual to the $n+1$ hyperplanes are located. Since any nonsingular curve of degree r is projectively equivalent to the following curve (the Veronese embedding of P^1):

$$t_0=(s_0)^r,\ t_1=(s_0)^{r-1}s_1,\dots,t_{r-1}=s_0(s_1)^{r-1},\ t_r=(s_1)^r$$

parametrized by $s_0 : s_1 \in P^1$, the manifold Q can be parametrized by the configuration space $X(2,n+1)$ of $n+1$ points on the projective line as follows:

$$\iota : X(2,n+1)\ni \xi = \begin{pmatrix} \xi_{00} & \cdots \xi_{0n} \\ \xi_{10} & \cdots \xi_{1n} \end{pmatrix} \mapsto \begin{pmatrix} (-\xi_{00})^r & \cdots & (-\xi_{0n})^r \\ (-\xi_{00})^{r-1}\xi_{10} & \cdots & (-\xi_{0n})^{r-1}\xi_{1n} \\ \vdots & \cdots & \vdots \\ -\xi_{00}\xi_{10}^{r-1} & \cdots & -\xi_{0n}\xi_{1n}^{r-1} \\ \xi_{10}^r & \cdots & \xi_{1n}^r \end{pmatrix} \in Q.$$

Without loss of generality, in what follows, we assume for $x=(x_{ij})\in X$ that

$$x_{00}=1, x_{10}=\cdots=x_{r0}=0;$$

the 0-th hyperplane is given by $t_0=0$, which we regard as a hyperplane at infinity H_0. The remaining r hyperplanes $H_j (1\le j\le n)$ in the complex affine space T with coordinate $t=(t_1,\dots,t_r)$ is defined by

$$L_j=\sum_{i=0}^{r} x_{ij}t_i, \quad t_0=1, \quad (1\le j\le n);$$

thus a point of X is now expressed by

$$x=(x_{ij}), \quad 0\le i\le r,\ 1\le j\le n.$$

Similarly, we assume for $\xi \in X(2, n+1)$ that

$$\xi_{00} = 1, \quad \xi_{10} = 0,$$

in other words, the 0-th point is given by $s_0 = 0$, which we regard as a point at infinity and think the remaining r points $\xi_j \, (1 \le j \le n)$ be in the complex affine line S with coordinate $s = s_1/s_0$; thus a point of $X(2, n+1)$ is now expressed by

$$\xi = (\xi_1, \dots, \xi_n).$$

Note that these conventions agree with the isomorphism $\iota : X(2, n+1) \to Q$, i.e.

$$(\xi_0 = \infty, \xi_1, \dots, \xi_n) \quad \mapsto \quad \begin{pmatrix} 1 & (-\xi_1)^r & \cdots & (-\xi_n)^r \\ 0 & (-\xi_1)^{r-1} & \cdots & (-\xi_n)^{r-1} \\ \vdots & \vdots & \cdots & \cdots \\ 0 & -\xi_1 & \cdots & -\xi_n \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Define $X_{\mathbf{R}}$ to be the real submanifold of X consisting of the points that can be represented by real matrices $x = (x_{ij})$, define $X_{\mathbf{R}}(2, n+1)$ similarly and put

$$Q_{\mathbf{R}} = Q \cap X_{\mathbf{R}}.$$

Then the restriction of the above map ι gives the isomorphism between $X_{\mathbf{R}}(2, n+1)$ and $Q_{\mathbf{R}}$. Similar convention will be applied also to the spaces T and S in order to define $T_{\mathbf{R}}$ and $S_{\mathbf{R}}$.

We choose a point $\dot{x}$ on $Q_{\mathbf{R}}$ as follows that will be fixed throughout the paper. Choose real numbers $\dot{\xi}_1, \dots, \dot{\xi}_n$ such that

$$\dot{\xi}_1 < \dots < \dot{\xi}_j < \dots < \dot{\xi}_n;$$

the point $\dot{\xi} = (\dot{\xi}_1, \dots, \dot{\xi}_n)$ lies in $X(2, n+1)$ and the point $\dot{x} = \iota(\dot{\xi})$ represents $n+1$ hyperplanes (cf. Picture 1):

$$H_0 = \text{the hyperplane at infinity,}$$

$$\dot{H}_j = H_j(\dot{x}), \quad 1 \le j \le n.$$

Note that each $\dot{H}_j$ is defined by the linear form

$$L_j(\dot{x}) := t_r + (-\dot{\xi}_j) t_{r-1} + \cdots + (-\dot{\xi}_j)^{r-1} t_1 + (-\dot{\xi}_j)^r.$$

2. Twisted cycles and a basis of solutions

Let $x \in X_{\mathbb{R}}$ be a point near to $\overset{\bullet}{x}$, and $T_{\mathbb{R}}$ the real affine space with coordinates $t = (t_1, \ldots, t_r)$; we fix an orientation of $T_{\mathbb{R}}$ once and for all. The complement of the n hyperplanes $\cup_j H_j$ in $T_{\mathbb{R}}$ has $\binom{n-1}{r}$ relatively compact components, which we label as follows: For

$$P = (p_1, \ldots, p_r), \quad 1 \leq p_1 < \cdots < p_r \leq n-1,$$

define

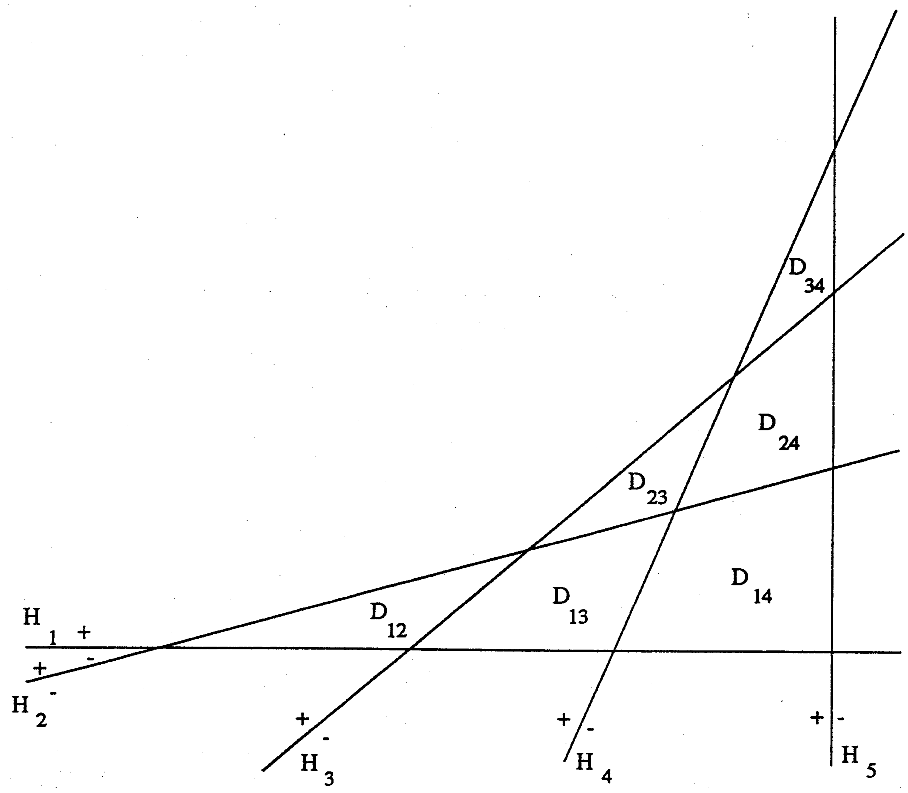
$$D'_P = D'_P(x) = \{t \in T_{\mathbb{R}} \mid (-1)^{P(j)} L_j(x) > 0\},$$

where $P(j)$ is defined after [Ter] by

$$P(j) := \text{Cardinality of } \{i \mid p_i < j\},$$

and

$$L_j(x) = \sum_{i=0}^{r} x_{ij} t_i.$$



Picture 1 : $r = 2, \quad n = 5$

Consider the following multi-valued r-form Ω on T:

$$\Omega = \Omega(x) := \prod_j L_j^{\alpha_j - 1} dt_1 \wedge \cdots \wedge dt_r.$$

On the domain D'_P, we assign arguments of L_j as

$$\arg L_j = -P(j)\pi, \quad 1 \leq j \leq n;$$

the domain D'_P, with the induced orientation as a domain in $T_{\mathbb{R}}$, together with the branch of Ω on it thus defined will be called the (twisted) cycle D_P. Put

$$u_P = u_P(x) = \int_{D_P} \Omega;$$

these define, by analytic continuation, holomorphic functions in $x = (x_{ij})$ around $\overset{\bullet}{x}$. It is shown in [Kit] that they form a basis of solutions of the hypergeometric differential equation $E(r+1, n+1; \alpha)$ if $\alpha_0, \ldots, \alpha_n \notin \mathbb{Z}$.

3. Circuit matrix $M(1, \ldots, r+1; \alpha)$

The domain $D'_{(1,\ldots,r)}(\overset{\bullet}{x})$ of $T_{\mathbb{R}}$ is a simplex bounded by $r+1$ hyperplanes $\overset{\bullet}{H}_1, \ldots, \overset{\bullet}{H}_r$ and $\overset{\bullet}{H}_{r+1}$. In this section, we study the circuit matrix $M(1, \ldots, r+1; \alpha)$ of the system $(D_P)_P$ or of the system $(u_P)_P$ relative to the loop $\rho(1, \ldots, r+1)$. The loop is described as follows: make a parallel displacement of the hyperplane $H_{r+1}(x)$ in $T_{\mathbb{R}}$ so that the simplex becomes small, let the hyperplane go once around in the complex space T the intersection point $\overset{\bullet}{H}_1 \cap \cdots \cap \overset{\bullet}{H}_r$ in the positive sense, and let it go back; during the whole journey, we keep $H_{r+1}(x)$ always parallel to $\overset{\bullet}{H}_{r+1}$. A similar moving of another hyperplane around the intersection point of the remaining r hyperplanes defines a loop homotopic to $\rho(1, \ldots, r+1)$.

Proposition (cf. [Aom], [Pha].) *The analytic continuation along the loop $\rho(1, \ldots, r+1)$ induces the transformation $M(1, \ldots, r+1; \alpha)$ of the functions u_P as follows:*

$$u_k \mapsto u_k + (-1)^{r-k} e(\alpha_{k+1} + \cdots + \alpha_{r+1})(1 - e(\alpha_1 + \cdots + \alpha_k))u_{r+1}, \quad 1 \leq k \leq r,$$
$$u_{r+1} \mapsto e(\alpha_1 + \cdots + \alpha_{r+1})u_{r+1},$$

where $e(\cdot) = \exp(2\pi i \cdot)$ and

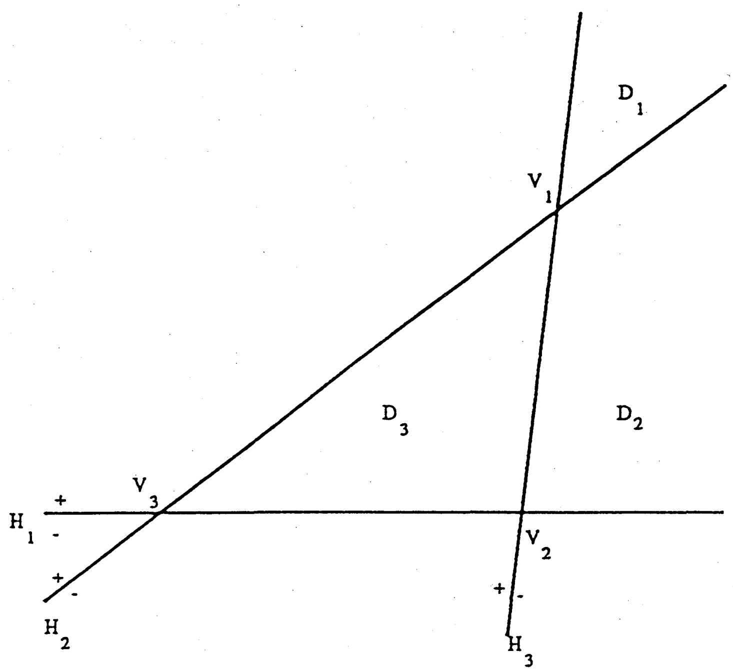
$$u_k := u_{(1,\ldots,k-1,k+1,\ldots,r+1)}, \quad 1 \leq k \leq r+1.$$

The function u_P does not change for other P. The equivalent statements for cycles are as follows:

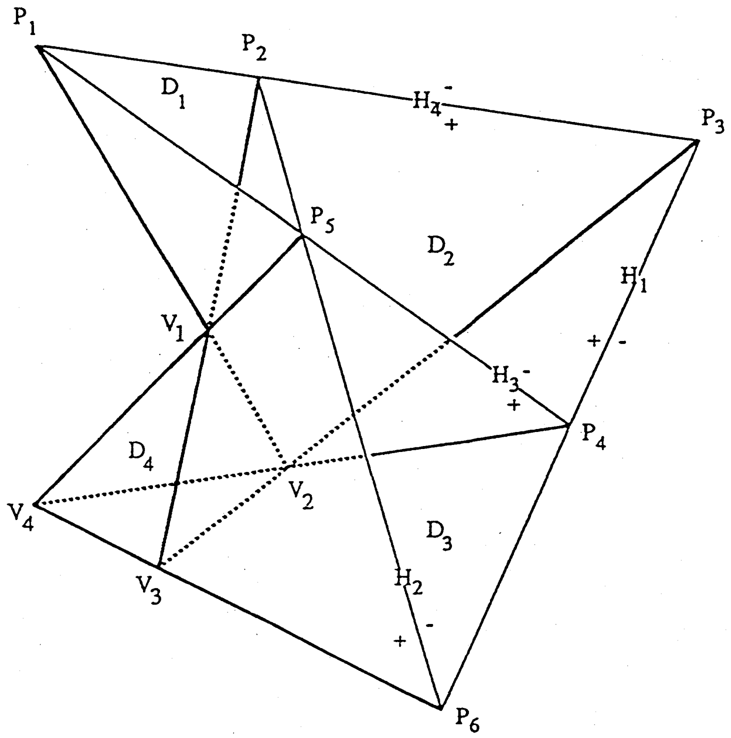
$$D_k \mapsto D_k + (-1)^{r-k} e(\alpha_{k+1} + \cdots + \alpha_{r+1})(1 - e(\alpha_1 + \cdots + \alpha_k))D_{r+1}, \quad 1 \leq k \leq r,$$
$$D_{r+1} \mapsto e(\alpha_1 + \cdots + \alpha_{r+1})D_{r+1},$$

where

$$D_k := D_{(1,\ldots,k-1,k+1,\ldots,r+1)}, \quad 1 \leq k \leq r+1.$$



Picture 2: $r = 2$



Section by a generic hyperplane $P_1P_2P_3P_4P_5P_6$ which we regard as H_5 is added in order to show the hyperplanes $H_1,\ldots,H_4$ and the domains $D'_2,\ldots,D'_4$: $D'_1=(V_1,P_1,P_2,P_5)$ a simplex, $D'_2=(V_1,V_2,P_4,P_3,P_2,P_5)$ a polytope, $D'_3=(V_1,V_2,V_3,P_6,P_4,P_5)$ a polytope, $D'_4=(V_1,V_2,V_3,V_4)$ a simplex.

Picture 3: $r = 3$

Proof. We apply the above convention also to the domains D'_P. The real domain $D'_{r+1} = D'_{(1,\dots,r)}$ is a simplex bounded by $r+1$ hyperplanes $\overset{\bullet}{H}_1, \dots, \overset{\bullet}{H}_{r+1}$. We name its $r+1$ vertices:

$$V_k = \overset{\bullet}{H}_1 \cap \cdots \cap \overset{\bullet}{H}_{k-1} \cap \overset{\bullet}{H}_{k+1} \cap \cdots \cap \overset{\bullet}{H}_{r+1}, \quad 1 \le k \le r+1.$$

There are exactly r domains D'_P which touch the simplex D'_{r+1}; in fact

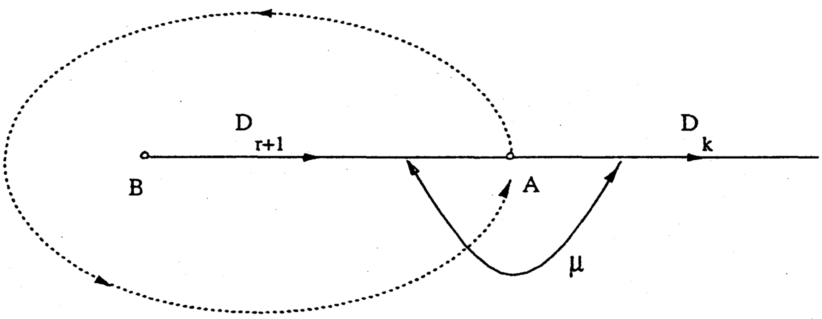
$$D'_1, \dots, D'_k, \dots, D'_r$$

touch the simplex D'_{r+1} along the following faces:

$$V_1, \dots, (V_1, \dots, V_k), \dots, (V_1, \dots, V_r),$$

respectively, where $(V_1, \dots, V_k)$ denotes the $(k-1)$-simplex with vertices $V_1, \dots, V_k$. It is obvious that by the move of the arrangement along $\rho(1, \dots, r+1)$ only $D_1, \dots, D_{r+1}$ among the $\binom{n-1}{r}$ cycles D_P would change.

In order to study the change of D_k, we consider the complex line l passing through a point A in the simplex $(V_1, \dots, V_k)$ and a point B in the complementary simplex $(V_{k+1}, \dots, V_{r+1})$ (see Picture 6). Picture 4 shows the line l as well as the points A, B and the two segments $l \cap D'_{r+1}$ and $l \cap D'_k$.

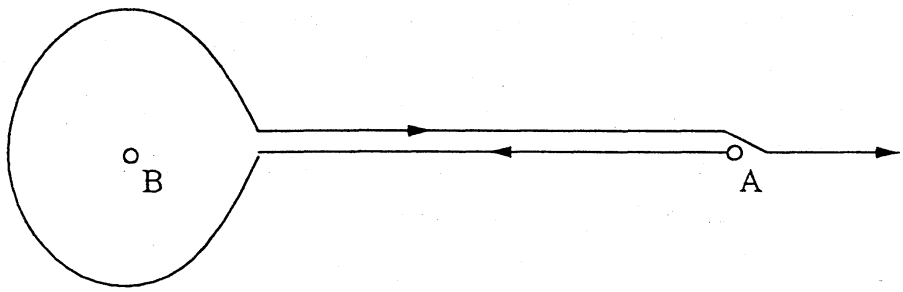


Picture 4: Line l

Our assignment of arguments of L_j says that

$$\begin{array}{rlll} \arg L_{k+1} = & -k\pi & \text{on } D'_{r+1}, & -(k-1)\pi \quad \text{on } D'_k, \\ \vdots & \vdots & & \vdots \\ \arg L_{r+1} = & -r\pi & \text{on } D'_{r+1}, & -(r-1)\pi \quad \text{on } D'_k. \end{array}$$

Therefore, for each m $(k+1 \le m \le r+1)$, the power function $L_m^{\alpha_m - 1}$ defined on D'_{r+1} and that defined on D'_k are analytic continuations of each other along a path μ in the *lower* half plane of the line l. According to the move along $\rho(1, \dots, r+1)$, the point A goes once around the point B in the positive direction (see the doted curve in Picture 4); this causes such a change of the segment $l \cap D'_k$ as shown in Picture 5.



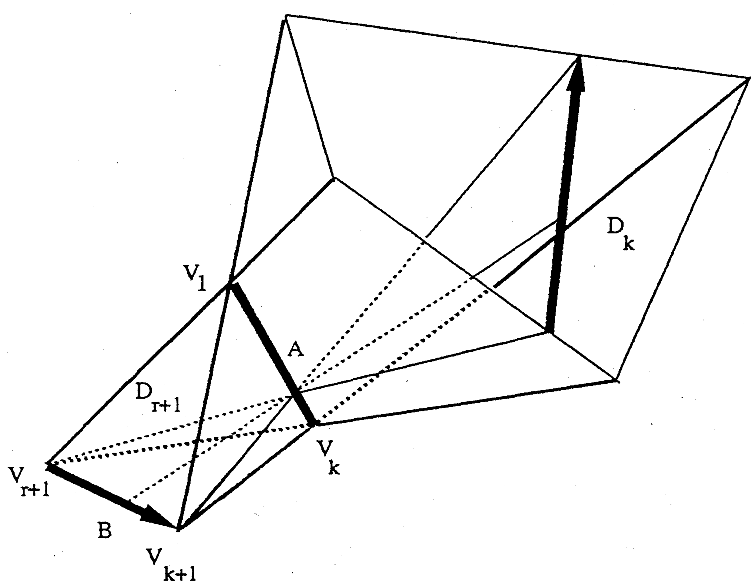
Picture 5 : The transformed curve

Since the transformed curve of $l \cap D'_k$ passes *above* the point A and goes around the point B, we have

$$l \cap D_k \looparrowright l \cap D_k + e(\alpha_{k+1} + \cdots + \alpha_{r+1})(1 - e(\alpha_1 + \cdots + \alpha_k)) l \cap D_{r+1} \quad 1 \leq k \leq r.$$

When the point A is fixed and the point B moves in $(V_{k+1}, \ldots, V_{r+1})$, we consider a map sending A to the point antipodal of A relative to B (see Picture 6); the map is orientation preserving or reversing if the dimension of the simplex $(V_{k+1}, \ldots, V_{r+1})$, which is equal to $r - k$, is even or odd, respectively. When the point B is fixed and the point A moves in $(V_1, \ldots, V_k)$ in some direction, then $l \cap D'_{r+1}$ and $l \cap D'_k$ move in the same direction. Since D'_{r+1} is the join of two simplices $(V_1, \ldots, V_k)$ and $(V_{k+1}, \ldots, V_{r+1})$, we have

$$D_k \looparrowright D_k + (-1)^{r-k} e(\alpha_{k+1} + \cdots + \alpha_{r+1})(1 - e(\alpha_1 + \cdots + \alpha_k)) D_{r+1}, \quad 1 \leq k \leq r.$$



Picture 6

In the course of this procedure the segment $l \cap D'_{r+1}$ turns around the point B as well as the point A. Thus we have

$$D_{r+1} \rightsquigarrow e(\alpha_1 + \cdots + \alpha_{r+1}) D_{r+1}.$$

The proof is now complete.

4. Relation between $E(r+1, n+1; \alpha)$ and $E(2, n+1; \alpha')$ ([Ter])

In this section, when x is on Q, we show that the r-dimensional integral $u_P(x)$ is an r-determinant of 1-dimensional integrals. Let $\xi_1, \dots, \xi_n$ be real points on the line S sufficiently near to $\dot{\xi}_1, \dots, \dot{\xi}_n$ so that

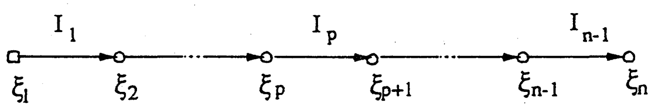
$$\xi_1 < \ldots < \xi_j < \ldots < \xi_n.$$

Let us define, for each q $(1 \le q \le n-1)$, a 1-form

$$\omega_q = \omega_q(\xi) := \prod_j (s - \xi_j)^{\alpha_j - 1} s^{q-1} ds,$$

which is single-valued in the lower half plane and continued analytically to the whole space S, and, for p $(1 \le p \le n-1)$, segments

$$I'_p := \{ s \in S_{\mathbb{R}} \mid \xi_p < s < \xi_{p+1} \}.$$



The form ω_q is defined in the lower half plane

Picture 7

On the segment I'_p, we assign arguments of $s - \xi_j$ as follows

$$\arg(s - \xi_j) = \begin{cases} 0, & \text{if } 1 \le j \le p; \\ -\pi, & \text{if } p+1 \le j \le n; \end{cases}$$

the segment I'_p, with the orientation indicated in Picture 7, together with the branch of ω_q thus defined will be called the (twisted) cycle I_p. Put

$$a_{pq}(\xi) := \int_{I_p} \omega_q$$

and, for

$$P = (p_1, \dots, p_r), \quad 1 \le p_1 < \ldots < p_r \le n-1,$$

put

$$A_P(\xi) := \det(a_{p_\mu \nu})^r_{\mu,\nu=1};$$

these define, by analytic continuation, holomorphic functions of $\xi = (\xi_1, \ldots, \xi_n) \in X(2, n+1)$ around $\overset{\bullet}{\xi} = (\overset{\bullet}{\xi}_1, \ldots, \overset{\bullet}{\xi}_n)$.

Let $x = (x_{ij})$ be a point on $Q \subset X$ corresponding to the point $\xi \in X(2, n+1)$, i.e., $x = \iota(\xi)$.

Proposition ([Ter]). *If $x \in Q \subset X$ is related with $\xi \in X(2, n+1)$ as above, we have $u_P(x) = A_P(\xi)$, i.e.,*

$$\int_{D_P} \prod_{j=1}^n L_j(x)^{\alpha_j - 1} dt_1 \wedge \cdots \wedge dt_r = \det\Big(\int_{I_{p_\mu}} \prod_{j=1}^n (s - \xi_j)^{\alpha_j - 1} s^{\nu - 1} ds\Big)^r_{\mu,\nu=1}$$

where

$$L_j(x) = t_r + (-\xi_j)t_{r-1} + \cdots + (-\xi_j)^{r-1} t_1 + (-\xi_j)^r.$$

Idea of the proof: Let $S^{(i)} (1 \leq i \leq r)$ with coordinates $s^{(i)}$ be r copies of the line S, and S^r be the product of these. Define a map $\phi : S^r \to T, \phi(s^{(i)}) = (t_i)$, by the following relation:

$$\prod_{i=1}^r (s^{(i)} - z) = t_r + (-z)t_{r-1} + \ldots + (-z)^r.$$

Then we have

$$\phi^* \Omega = \sum_{\sigma \in \mathfrak{S}_r} \omega_1^{(\sigma(1))} \wedge \cdots \wedge \omega_q^{(\sigma(r))},$$

where $\omega_q^{(i)}$ is the pull back of ω_q under the projection of C^r to $C^{(i)}$:

$$\omega_q^{(i)} = \prod_j (s^{(i)} - \xi_j)^{\alpha_j - 1} (s^{(i)})^{q-1} ds^{(i)},$$

and $\mathfrak{S}_r$ is the symmetric group in r letters. The cycles on S^r and on T are related as follows:

$$\phi(I_{p_1}^{(1)} \times \cdots \times I_{p_r}^{(r)}) = D_P,$$

where $P = (p_1, \ldots, p_r)$ and $I_{p_k}^{(k)}$ is a cycle on $S^{(k)}$ which is the copy of the cycle I_{p_k} on S. These two assertions can be checked by a direct computation. By virtue of these facts the proposition can be readily proved.

5. Action of the braid group B_{n+1} on a collection of solutions of $E(2, n+1; \alpha)$

For notational symplicity, we use ω and a_p in place of ω_1 and a_{p1}:

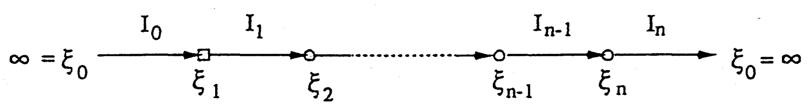
$$a_p(\xi) = \int_{I_p} \omega(\xi) = \int_{I_p} \prod_{j=1}^n (s - \xi_j)^{\alpha_j - 1} ds, \quad (1 \leq p \leq n-1),$$

which are functions of ξ around $\overset{\bullet}{\xi}$. Recall that the arguments of $s - \xi_j$ were so assigned that the form ω is defined in the *lower* half s-plane (see Picture 8); keeping this assignment, we write the above integrals as follows:

$$a_p(\xi) = \int_{\xi_p}^{\xi_{p+1}} \omega(\xi).$$

To recover symmetry, we re-introduce the point $\overset{\bullet}{\xi}_0 = \infty$ and a real point ξ_0 near $\overset{\bullet}{\xi}_0$, and define a_0 and a_n as follows:

$$a_0(\xi) = \int_{\xi_0}^{\xi_1} \omega(\xi), \quad a_n(\xi) = \int_{\xi_n}^{\xi_0} \omega(\xi).$$



The form ω is defined in the lower half plane

Picture 8

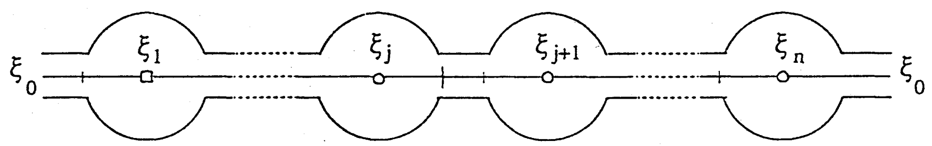
Lemma. *Among these* $n+1$ *functions* $a_j = a_j(\xi)$ *defined around* $\overset{\bullet}{\xi}$, *hold two linear relations:*

$$\sum_{j=0}^{n} a_j = 0,$$

$$\sum_{j=0}^{n} e(-\alpha_0 - \cdots - \alpha_j) a_j = 0,$$

where $e(\cdot) = \exp(2\pi i \cdot)$.

Proof: One has only to integrate ω along the curves shown in Picture 9.

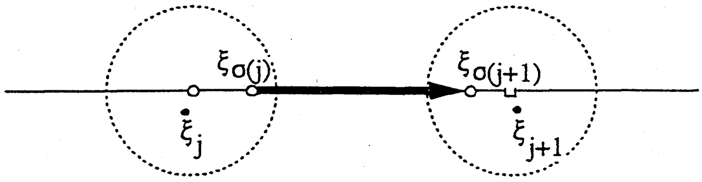


Picture 9

Let $\mathfrak{S}_{n+1}$ be the permutation group in $n+1$ letters $0, 1, \ldots, n$. For $\sigma \in \mathfrak{S}_{n+1}$, we define $a_j^\sigma (0 \leq j \leq n)$ by

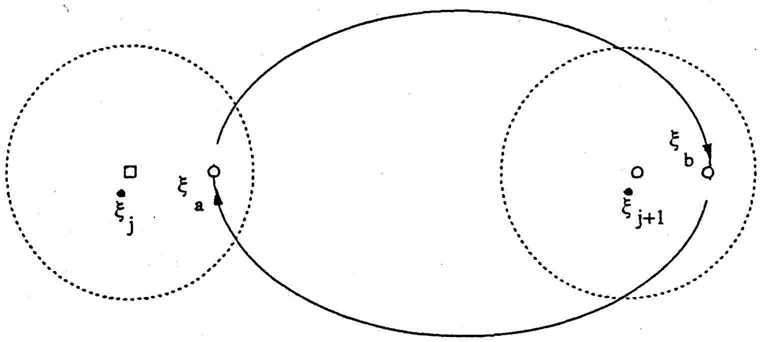
$$a_j^\sigma(\xi) = \int_{\xi_{\sigma^{-1}(j)}}^{\xi_{\sigma^{-1}(j+1)}} \omega(\xi), \quad 0 \leq j \leq n,$$

where, for each k, $\xi_{\sigma(k)}$ is supposed to be near to $\overset{\bullet}{\xi}_k$; they are functions in $(\xi_0, \ldots, \xi_n)$ around $(\xi_{\sigma^{-1}(0)}, \ldots, \xi_{\sigma^{-1}(n)})$. Note that $a_j^\sigma = a_j$ when σ is the identity and that, when σ is not the identity, the domain of definition of $\{a_j\}_j$ and of $\{a_j^\sigma\}_j$ are disjoint (see Picture 10). Let A^σ be the linear span of $\{a_j^\sigma\}_j$.



Picture 10

Let B_{n+1} be the braid group generated by the exchange s_p of the point near $\overset{\bullet}{\xi}_p$ and the point near $\overset{\bullet}{\xi}_{p+1}$ $(0 \leq p \leq n-1)$ as is indicated in Picture 11.



Picture 11

Let $\rho : B_{n+1} \longrightarrow \mathfrak{S}_{n+1}$ be the natural homomorphism (of which kernel is the colored braid group) defined by:

$$\rho : B_{n+1} \ni s_p \mapsto \sigma_p = (p, p+1) \in \mathfrak{S}_{n+1}.$$

Every element $s \in B_{n+1}$ causes, by the analytic continuation along the path shown in Picture 11, a linear isomorphism $N(s) = N(s; \alpha) : A \longrightarrow A^{\rho(s)}$. Generally, for any $\sigma \in \mathfrak{S}_{n+1}$, we have an isomorphism

$$N^\sigma(s) = N^\sigma(s; \alpha) : A^\sigma \longrightarrow A^{\sigma\rho(s)}.$$

Notice that, by definition, we have

$$N^\sigma(ss') = N^\sigma(s) N^{\sigma\rho(s)}(s');$$

this formula will be quoted by the name of *1-cocycle property*. Actual transformations are given in the following proposition in terms of the generators of the braid group and the functions a_j^σ; although the spaces A^σ are $(n-1)$-dimensional, we make use of $n+1$ functions $a_j^\sigma (0 \le j \le n)$ in order to make the following formulae simple and symmetric.

Let the group $\mathfrak{S}_{n+1}$ act on the parameter α as follows:

$$(\alpha^\sigma)_j = \alpha_{\sigma^{-1}(j)}, \quad 0 \le j \le n.$$

Proposition. *For each generator $s_p (0 \le p \le n-1)$ of B_{n+1}, the action of $N^\sigma(s_p;\alpha)$ is given as follows by the use of functions $\{a_j^\sigma\}$ and $\{a_k^{\sigma\sigma_p}\}, 0 \le k \le n$:*

$$\begin{aligned}
a_{p-1}^\sigma &\curvearrowright a_{p-1}^{\sigma\sigma_p} + \frac{1}{(c^\sigma)_{p+1}} a_p^{\sigma\sigma_p}, \\
a_p^\sigma &\curvearrowright \frac{-1}{(c^\sigma)_{p+1}} a_p^{\sigma\sigma_p}, \\
a_{p+1}^\sigma &\curvearrowright a_p^{\sigma\sigma_p} + a_{p+1}^{\sigma\sigma_p}, \\
a_j^\sigma &\curvearrowright a_j^{\sigma\sigma_p}, \quad (j \ne p-1, p, p+1),
\end{aligned}$$

where a_{-1}^σ should be read as a_n^σ, and

$$(c^\sigma)_j := e((\alpha^\sigma)_j) = \exp\{2\pi i (\alpha^\sigma)_j\}.$$

By virtue of Lemma, we get the matrix representation of $N^\sigma(s_p;\alpha)$ by using the bases $\{a_1^\sigma, \ldots, a_n^\sigma\}$ and $\{a_1^{\sigma\sigma_p}, \ldots, a_n^{\sigma\sigma_p}\}$, also denoted by $N^\sigma(s_p;\alpha)$:

$$s_p : {}^t\{a_1^\sigma, \ldots, a_n^\sigma\} \curvearrowright N^\sigma(s_p;\alpha)^t\{a_1^{\sigma\sigma_p}, \ldots, a_n^{\sigma\sigma_p}\}.$$

Remark. As matrices, we have

$$N^\sigma(s_p;\alpha) = N(s_p;\alpha^\sigma).$$

Their determinants do not vanish for any α.

Now we can know how the $\binom{n-1}{r}$ functions

$$A_P(\xi) = \det(a_{p_\mu \nu})_{\mu,\nu=1}^r$$

change: Since the forms ω_q have the same monodromy property as that of $\omega = \omega_1$, the change of $\{A_P\}_P$ can be expressed by the r-exterior product $\wedge^r N^\sigma(s_p;\alpha)$ of $N^\sigma(s_p;\alpha)$; arranging P in the lexicographic order, we denote by $W^\sigma(s_p;\alpha)$ the corresponding matrix:

$$W^\sigma(s_p;\alpha) = \wedge^r N^\sigma(s_p;\alpha) : \wedge^r A_P^\sigma \longrightarrow \wedge^r A_P^{\sigma\rho(s_p)}.$$

6. Generators

Let $u_P(x)$ be the functions around $\overset{\bullet}{x} \in X$ defined in Section 2, and $u_P^\sigma(x)$ the functions around the point of X corresponding to $\overset{\bullet}{\xi}{}^\sigma = (\overset{\bullet}{\xi}_{\sigma(0)}, \dots, \overset{\bullet}{\xi}_{\sigma(n+1)}) \in X(2, n+1)$ defined exactly the same way with the parameter α^σ; we arrange them in the lexicographic order in columns and denote them by u and u^σ. For

$$J = (j_1, \dots, j_{r+1}), \quad 1 \leq j_1 < \cdots < j_{r+1} \leq n+1$$

($n+1$ should be interpreted as 0), let $M(J;\alpha)$ be the circuit matrix with respect to u corresponding to the loop $\rho(J)$:

$$\rho(J) : u \mapsto M(J;\alpha)u;$$

similarly, let $M^\sigma(J;\alpha)$ be the matrix with respect to u^σ. Notice that as matrices we have

$$M^\sigma(J;\alpha) = M(J;\alpha^\sigma).$$

The matrix $M(1, \dots, r+1;\alpha)$, which is holomorphic in α, is given in the proposition in Section 3. Since

$$u_P(x) = A_P(\xi), \quad x = \iota(\xi)$$

(Proposition in Section 4), by the argument in the preceding section, the other $M(J;\alpha)$ can be obtained by the recurrence formula in the following theorem.

Theorem. *Let $\alpha_0, \dots, \alpha_n$ be complex numbers such that*

$$\alpha_j \notin \mathbb{Z}, \quad \alpha_0 + \cdots + \alpha_n = n - r.$$

The generators $M(J;\alpha)$ of the monodromy group of the hypergeometric differential equation $E(r+1, n+1;\alpha)$ with respect to the fundamental solutions $\{u_P\}_P$ are given by the following recurrence formula with initial datum $M(1, \dots, r+1;\alpha)$. If $j_k + 1 < j_{k+1}$ or $j_{r+1} + 1 \leq n+1$, then

$$M(J + \varepsilon_k;\alpha) = W(s_{j_k};\alpha) M^{\sigma_{j_k}}(J;\alpha) W(s_{j_k};\alpha)^{-1},$$

where $J = (j_1, \dots, j_{r+1})$ and $\varepsilon_k = (0, \dots, 0, \overset{k}{1}, 0, \dots, 0)$. The matices $M(J;\alpha)$ are holomorphic and invertible for any value of α.

Remark. Compatibility of this recurrence formula can be derived from the 1-cocycle property of $W^\sigma(s;\alpha)$. Notice that if both $J + \varepsilon_k$ and $J + \varepsilon_l$ belong to the due range of parameters, we must have $|j_k - j_l| \geq 2$ and so s_{j_k} and s_{j_l} are commutative. Then

$$\begin{aligned}
M((J + \varepsilon_k) + \varepsilon_l;\alpha) &= W(s_{j_l};\alpha) M^{\sigma_{j_l}}(J + \varepsilon_k;\alpha) W(s_{j_l};\alpha)^{-1} \\
&= W(s_{j_l};\alpha) W^{\sigma_{j_l}}(s_{j_k};\alpha) M^{\sigma_{j_k}\sigma_{j_l}}(J;\alpha) W^{\sigma_{j_l}}(s_{j_k};\alpha)^{-1} W(s_{j_l};\alpha)^{-1} \\
&= W(s_{j_l} s_{j_k};\alpha) M^{\sigma_{j_k}\sigma_{j_l}}(J;\alpha) W(s_{j_l} s_{j_k};\alpha)^{-1} \\
&= W(s_{j_k} s_{j_l};\alpha) M^{\sigma_{j_l}\sigma_{j_k}}(J;\alpha) W(s_{j_k} s_{j_l};\alpha)^{-1} \\
&= M((J + \varepsilon_l) + \varepsilon_k;\alpha).
\end{aligned}$$

Remark. By virtue of the remark above, the actual computation can be done in an economic way as follows: Let $J' = (j'_1, \dots, j'_{r+1})$ be the next one of $J = (j_1, \dots, j_{r+1})$ with respect to the lexicographic order; there exists k $(1 \le k \le r+1)$ such that

$$j'_1 = j_1, \dots, j'_{k-1} = j_{k-1},\ j'_k = j_k + 1.$$

Then $M(J'; \alpha)$ is given by

$$W(s_{j_k}; \alpha) M^{\sigma_{j_k}}(j_1, \dots, j_k, j'_{k+1}, \dots, j'_{r+1}; \alpha) W(s_{j_k}; \alpha)^{-1}.$$

Remark. The unique eigenvalue of the matrix $M(j_1, \dots, j_{r+1}; \alpha)$ which is not 1 is $e(\alpha_{j_1} + \dots + \alpha_{j_{r+1}})$ of multiplicity 1. Each matrix $M(j_1, \dots, j_{r+1}; \alpha)$ can be written by use of row $\binom{n-1}{r}$-vectors

$$a(j_1, \dots, j_{r+1}) \quad \text{and} \quad b(j_1, \dots, j_{r+1})$$

in the form:

$$M(j_1, \dots, j_{r+1}; \alpha) = Identity - {}^t a(j_1, \dots, j_{r+1}) b(j_1, \dots, j_{r+1}).$$

Example. $r = 2,\ n = 5$. We have

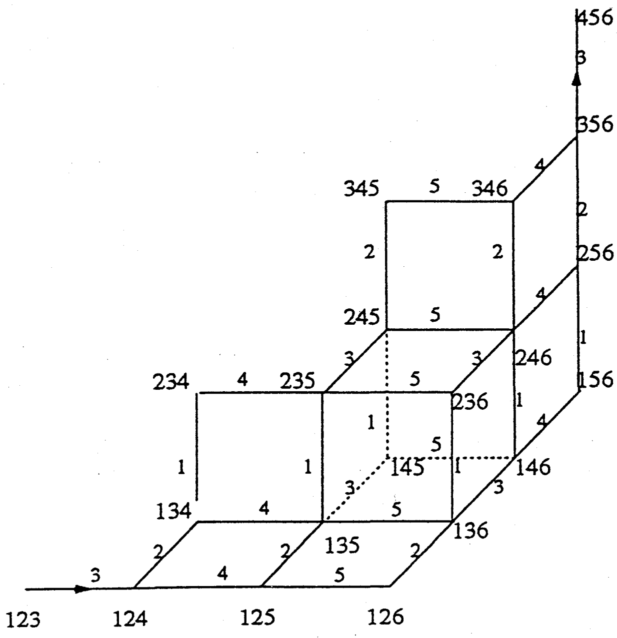
$$M(1,2,3;\alpha) = \begin{pmatrix} e(\alpha_1+\alpha_2+\alpha_3) & 0 & 0 & 0 & 0 & 0 \\ e(\alpha_3)(1-e(\alpha_1+\alpha_2)) & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -e(\alpha_2+\alpha_3)(1-e(\alpha_1)) & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$W(s_3,\alpha) = \begin{pmatrix} 1 & e(-\alpha_4) & 0 & 0 & 0 & 0 \\ 0 & -e(-\alpha_4) & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e(-\alpha_4) & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & e(-\alpha_4) \\ 0 & 0 & 0 & 0 & 0 & -e(-\alpha_4) \end{pmatrix},$$

$$M(1,2,4;\alpha) = W(s_3,\alpha) M(1,2,3;\alpha_0,\alpha_1,\alpha_2,\alpha_4,\alpha_3,\alpha_5) W(s_3,\alpha)^{-1}$$

$$= \begin{pmatrix} 1-e(\alpha_1+\alpha_2)(1-e(\alpha_4)) & -e(\alpha_1+\alpha_2)(1-e(\alpha_4)) & 0 & 0 & 0 & 0 \\ e(\alpha_1+\alpha_2)-1 & e(\alpha_1+\alpha_2) & 0 & 0 & 0 & 0 \\ e(\alpha_4)(1-e(\alpha_1+\alpha_2)) & e(\alpha_4)(1-e(\alpha_1+\alpha_2)) & 1 & 0 & 0 & 0 \\ e(\alpha_2)(1-e(\alpha_1)) & e(\alpha_2)(1-e(\alpha_1)) & 0 & 1 & 0 & 0 \\ -e(\alpha_2+\alpha_4)(1-e(\alpha_1)) & -e(\alpha_2+\alpha_4)(1-e(\alpha_1)) & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this way, we successively get the others as is indicated in Picture 12.



Picture 12

The matrices $M(j_1,j_2,j_3)$ are given by the following vectors $a(j_1,j_2,j_3)$ and $b(j_1,j_2,j_3)$; the index 6 should be read as 0, and

$$c_k = e(\alpha_k) = \exp(2\pi i\alpha_k),\ 1 \le k \le 6.$$

$$\begin{array}{lllllll}
a(123) = \big(1-c_1c_2c_3, & -(1-c_1c_2)c_3, & 0, & (1-c_1)c_2c_3, & 0, & 0 & \big)\\
a(124) = \big((1-c_4)c_1c_2, & 1-c_1c_2, & -(1-c_1c_2)c_4, & -(1-c_1)c_2, & (1-c_1)c_2c_4, & 0 & \big)\\
a(125) = \big(c_1c_2(1-c_5), & 0, & 1-c_1c_2, & 0, & -(1-c_1)c_2, & 0 & \big)\\
a(126) = \big(1, & 0, & 0, & 0, & 0, & 0 & \big)\\
a(134) = \big(-(1-c_4)c_1, & (1-c_3c_4)c_1, & -(1-c_3)c_1c_4, & 1-c_1, & -(1-c_1)c_4, & (1-c_1)c_3c_4 & \big)\\
a(135) = \big(-(1-c_5)c_1, & (1-c_5)c_1c_3, & (1-c_3)c_1, & 0, & 1-c_1, & -(1-c_1)c_3 & \big)\\
a(136) = \big(-\frac{(1-c_2)}{c_2}, & \frac{(1-c_2)c_3}{c_2}, & 0, & 0, & 0, & 0 & \big)\\
a(145) = \big(0, & -(1-c_5)c_1, & (1-c_4c_5)c_1, & 0, & 0, & 1-c_1 & \big)\\
a(146) = \big(0, & 1, & -c_4, & 0, & 0, & 0 & \big)\\
a(156) = \big(0, & 0, & 1, & 0, & 0, & 0 & \big)\\
a(234) = \big(1-c_4, & -(1-c_3c_4), & (1-c_3)c_4, & 1-c_2c_3c_4, & -(1-c_2c_3)c_4, & (1-c_2)c_3c_4 & \big)\\
a(235) = \big(1-c_5, & -(1-c_5)c_3, & -(1-c_3), & (1-c_5)c_2c_3, & 1-c_2c_3, & -(1-c_2)c_3 & \big)
\end{array}$$

$$
\begin{aligned}
a(236) &= (1, \quad -c_3, \quad 0, \quad c_2c_3, \quad 0, \quad 0)\\
a(245) &= (0, \quad 1-c_5, \quad -(1-c_4c_5), \quad -(1-c_5)c_2, \quad (1-c_4c_5)c_2, \quad 1-c_2)\\
a(246) &= (0, \quad 1, \quad -c_4, \quad -c_2, \quad c_2c_4, \quad 0)\\
a(256) &= (0, \quad 0, \quad 1, \quad 0, \quad -c_2, \quad 0)\\
a(345) &= (0, \quad 0, \quad 0, \quad 1-c_5, \quad -(1-c_4c_5), \quad 1-c_3c_4c_5)\\
a(346) &= (0, \quad 0, \quad 0, \quad 1, \quad -c_4, \quad c_3c_4)\\
a(356) &= (0, \quad 0, \quad 0, \quad 0, \quad 1, \quad -c_3)\\
a(456) &= (0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 1)
\end{aligned}
$$

$$
\begin{aligned}
b(123) &= (1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0)\\
b(124) &= (1, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0)\\
b(125) &= (1, \quad 1, \quad 1, \quad 0, \quad 0, \quad 0)\\
b(126) &= \left(1-c_1c_2c_6, \quad \frac{(1-c_1c_2c_3c_6)}{c_3}, \quad -(1-c_5)c_1c_2c_6, \quad 0, \quad 0, \quad 0\right)\\
b(134) &= (0, \quad 1, \quad 0, \quad 1, \quad 0, \quad 0)\\
b(135) &= (0, \quad 1, \quad 1, \quad 1, \quad 1, \quad 0)\\
b(136) &= \left(1-c_2, \quad \frac{(1-c_1c_2c_3c_6)}{c_3}, \quad -(1-c_5)c_1c_2c_6, \quad \frac{(1-c_1c_2c_3c_6)}{c_3}, \quad -(1-c_5)c_1c_2c_6, \quad 0\right)\\
b(145) &= (0, \quad 0, \quad 1, \quad 0, \quad 1, \quad 1)\\
b(146) &= \left(-\frac{(1-c_2)}{c_2}, \quad 1-c_1c_4c_5c_6, \quad (1-c_5)c_1c_6, \quad -(1-c_3)c_1c_4c_5c_6, \quad (1-c_5)c_1c_6, \quad (1-c_5)c_1c_6\right)\\
b(156) &= \left(-\frac{(1-c_2)}{c_2}, \quad 1-c_1c_4c_5c_6, \quad 1-c_1c_5c_6, \quad -(1-c_3)c_1c_4c_5c_6, \quad -(1-c_3c_4)c_1c_5c_6, \quad -(1-c_4)c_1c_5c_6\right)\\
b(234) &= (0, \quad 0, \quad 0, \quad 1, \quad 0, \quad 0)\\
b(235) &= (0, \quad 0, \quad 0, \quad 1, \quad 1, \quad 0)\\
b(236) &= \left(-\frac{(1-c_1)}{c_1}, \quad 0, \quad 0, \quad -(1-c_4c_5)c_6, \quad -(1-c_5)c_6, \quad 0\right)\\
b(245) &= (0, \quad 0, \quad 0, \quad 0, \quad 1, \quad 1)\\
b(246) &= \left(-\frac{(1-c_1)}{c_1}, \quad -\frac{(1-c_1)}{c_1}, \quad 0, \quad (1-c_3)c_4c_5c_6, \quad -(1-c_5)c_6, \quad -(1-c_5)c_6\right)\\
b(256) &= \left(-\frac{(1-c_1)}{c_1}, \quad -\frac{(1-c_1)}{c_1}, \quad -\frac{(1-c_1)}{c_1}, \quad (1-c_3)c_4c_5c_6, \quad (1-c_3c_4)c_5c_6, \quad (1-c_4)c_5c_6\right)\\
b(345) &= (0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 1)\\
b(346) &= \left(0, \quad -\frac{(1-c_1)}{c_1}, \quad 0, \quad 1-c_3c_4c_5c_6, \quad 0, \quad -(1-c_5)c_6\right)\\
b(356) &= \left(0, \quad -\frac{(1-c_1)}{c_1}, \quad -\frac{(1-c_1)}{c_1}, \quad 1-c_3c_4c_5c_6, \quad 1-c_3c_4c_5c_6, \quad (1-c_4)c_5c_6\right)\\
b(456) &= \left(0, \quad 0, \quad -\frac{(1-c_1)}{c_1}, \quad 0, \quad -\frac{(1-c_1c_2)}{c_1c_2}, \quad 1-c_4c_5c_6\right)
\end{aligned}
$$

References

[Aom] K. Aomoto, On the structure of intrgrals of power products of linear functions, Sci. Papers, Coll. Gen. Education, Univ. Tokyo, **27** (1977) 49–61.

[Gel] I.M. Gelfand, General thory of hypergeometric functions, Soviet Math. Dokl. **33** (1986) 573–577.

[IKSY] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, *From Gauss to Painlevé*, 1991, Vieweg, Wiesbaden.

[Kit] M. Kita, On the Wronskian of the hypergeometric system $E(k, k+l)$ (preprint), 1991.

[MSY] K. Matsumoto, T. Sasaki and M. Yoshida, The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeomtric function of type (3,6), Internat. J. of Math. **3** (1992) 1–164.

[MSTY] K. Matsumoto, T. Sasaki,N. Takayama and M. Yoshida, Monodromy of the hypergeometric differential equation of type (k,n) II (preprint), 1992.

[Pha] F. Pham, *Introduction a l'étude topologique des singularités de Landau*, 1967, Gauthiers Villars.

[Ter] T. Terasoma, Exponential Kummer coverings and determinants of hypergeometric funtions (preprint), 1991.

[Yos] M. Yoshida, *Fuchsian differential equations*, 1987, Vieweg, Wiesbaden.